A generalized image principle for the wave equation with absorbing boundary conditions and applications to fourth order schemes

T. Ha-Duong, Patrick Joly

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

Rapports de Recherche

N° 1333

Programme 7
Calcul Scientifique,
Logiciels Numériques et Ingénierie Assistée
par Ordinateur

A GENERALIZED IMAGE PRINCIPLE FOR THE WAVE EQUATION WITH ABSORBING BOUNDARY CONDITIONS AND APPLICATIONS TO FOURTH ORDER SCHEMES.

T. HA DUONG
P. JOLY

Novembre 1990

A generalized image principle for the wave equation with absorbing boundary conditions and applications to fourth order schemes.

Un principe des images généralisées pour l'équation des ondes avec conditions aux limites absorbantes et applications aux schémas du quatrième ordre.

T. HA DUONG[1] and P. JOLY[2]

1 CMAP, Ecole Polytechnique, 91128 Palaiseau Cédex.

2 INRIA, Domaine de Voluceau, Rocquencourt, 78153 Le Chesnay Cédex.

Résumé :

Lorsqu'on utilise des schémas d'ordre élevé pour l'équation des ondes, d'ordre quatre par exemple, le traitement des conditions aux limites pose une difficulté spécifique puisqu'il nécessite plusieurs équations supplémentaires (pour les noeuds proches de la frontière) alors qu'on ne dispose que d'une condition aux limites scalaire. Dans le cas de bords parfaitement réfléchissants, c'est-à-dire pour les conditions de Neumann ou Dirichlet homogènes, cette difficulté peut être résolue par l'utilisation du principe bien connu des images qui permet d'étendre l'équation à l'extérieur du domaine de calcul par une symétrisation appropriée des données. Nous proposons dans ce papier une généralisation de ce principe au cas des conditions aux limites absorbantes. A l'aide d'un procédé de symétrisation nous sommes amenés à introduire une équation des ondes amortie avec un terme d'amortissement ponctuel. Le traitement de la condition aux limites est alors remplacé par celui d'une nouvelle équation dans tout l'espace. La justification théorique de cette approche est fondée sur de nouvelles estimations d'énergie vérifiées par la solution de l'équation des ondes avec conditions aux limites absorbantes d'ordre élevé et constitue une alternative au célèbre critère de Kreiss pour démontrer la stabilité du problème aux limites associé.

Abstract :

When one uses high order finite difference schemes for the wave equation, for instance fourth order schemes, the treatment of boundary conditions poses a real difficulty since one needs several additional equations (for the nodes close to the boundary), while only one single scalar boundary condition is available. In the case of perfectly reflecting boundary conditions, namely the homogeneous Neumann or Dirichlet conditions, this difficulty can be overcomed by the use of the well-known image principle which permits to extend the equation outside of the domain of calculation by an appropriate symmetrization of the data. We propose in this paper a generalization of this principle to the absorbing boundary conditions. Through a symmetrization process, we are led to introduce a damped wave equation with a damping term supported by the boundary. The treatment of the boundary condition is then replaced by the approximation of this new damped wave equation in the whole space. The theoretical justification of our approach is based on new energy estimates for the wave equation (when high order absorbing boundary conditions are used), and constitutes an alternative to the use of the well-known Kreiss criterion to prove the stability of the associated initial boundary value problem.

Mots-Clés :

Conditions aux limites absorbantes, principe des images, schémas d'ordre quatre.

Key Words :

Absorbing boundary conditions, principle of images, fourth order schemes.

0. INTRODUCTION - MOTIVATION

When one is interested in the simulation of wave propagation, especially acoustic waves, with the help of classical methods in numerical analysis - we think in particular of finite differences - one of the main drawbacks comes from the numerical dispersion which induces an error in the propagation velocities. To minimize these effects, a natural idea consists of using high accuracy numerical schemes, for instance fourth order schemes in space and time. Such schemes have already been treated extensively in the literature ([1], [2], [4], [12]) and their stability and dispersion properties are now very well known. The major obstacle to the use of these schemes is the treatment of boundary conditions. Indeed, one is led, in order to get a high accuracy, to use finite difference operators using more mesh points than it is a priori necessary to be consistent. For instance, if we consider the wave equation in dimension 1 :

(0.1) $$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

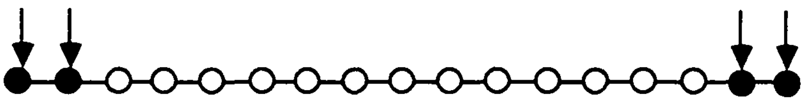
We shall use 5 points to approximate the second derivative in space :

(0.2) $$\frac{\partial^2 u}{\partial t^2} (x_j,t) \simeq \frac{4}{3h^2} (u_{j+1} - 2u_j + u_{j-1}) - \frac{1}{12h^2} (u_{j+2} - 2u_j + u_{j-2})$$

If the equation (0.1) is set in a segment $[0,L]$ and if this segment is discretized in N space steps :

(0.3) $$[0,L] = \bigcup_{j=0}^{N-1} [x_j,x_{j+1}] \qquad x_j = jh \ , \ h = \frac{L}{N}$$

The difference equation obtained in (0.2) can be used only for $N-3$ discretization nodes, as illustrated below :



One needs a priori 4 additional equations while one has only two boundary conditions, one at $x = 0$ and one at $x = L$.

When one considers Dirichlet or Neumann boundary conditions, this difficulty can be handled with the help of the well known image principle [3] which express that the initial boundary value problem is equivalent to a pure initial value problem on the whole line through a principle of multiple reflexion.

Thanks to this idea, even the points close to the boundary can be seen as interior points, the value at the extra fictive nodes being obtained by symmetry or antisymmetry depending on whether one considers Neumann or Dirichlet boundary conditions.

The situation is not so clear if one considers absorbing boundary conditions which are very important in practical situations. One possible approach consists of using extrapolation methods (see [1]). However the theoretical aspects of such an approach do not appear to us as completely under control and it seems that it remains a great part of arbitrariness in the precise use of such methods. Our purpose in this article is to propose an alternative to this approach based on the generalization to absorbing boundary conditions of the principle of images. Our article is organized as follows. In section 1, we describe the main idea in the case of the 1D wave equation. In section 2, we present the formal generalization of this idea to the wave equation in the half space for a general class of absorbing boundary conditions. Sections 3 and 4 are devoted to the complete mathematical justification of our approach for respectvely first order and second order absorbing boundary conditions. Finally, in section 5 we show how to use our new principle of images for the construction of stable approximations of absorbing boundary conditions coupled with fourth order schemes for the interior wave equation. Conclusions and perspectives of the present work are given in section 6.

1. THE BASIC PRINCIPLE IN THE 1D CASE

We consider the simple model problem of the wave equation on the half-line with the transparent boundary condition at $x = 0$

$$(1.1) \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad x < 0 \, , t > 0$$

$$(1.2) \qquad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad x = 0 \, , t > 0$$

If we prescribe initial conditions

$$(1.3) \qquad \begin{cases} u(x,0) = u_0(x) \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \end{cases}$$

the initial boundary value problem ((1.1),(1.2),(1.3)) is well posed. More precisely the energy of the solution u is decreasing. Indeed one has the identity :

$$(1.4) \qquad \frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^-}\left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^-}\left|\frac{\partial u}{\partial x}\right|^2 dx\right\} + \left|\frac{\partial u}{\partial t}(0,t)\right|^2 = 0$$

From (1.4), it is easy to deduce that, under the assumptions :

$$(1.5) \qquad (u_0,u_1) \in H^1(\mathbb{R}^-) \times L^2(\mathbb{R}^-)$$

Then the unique "weak" solution u of ((1.1),(1.2),(1.3)) satisfies :

$$(1.6) \qquad u \in C^1\left(\mathbb{R}^+ ; L^2(\mathbb{R}^-)\right) \cap C^0\left(\mathbb{R}^+ ; H^1(\mathbb{R}^-)\right)$$

Let us now consider the following question : can we define a problem set on the whole real line whose restriction to the half line $\mathbb{R}^-$ would coincide with the solution u of our model problem ?

If we replace the boundary condition (1.2) by either the Neumann or Dirichlet condition, the answer is very simple : it suffices to extend the initial conditions u_0 and u_1 by symmetry (for Neumann) or antisymmetry (for Dirichlet) and to solve the corresponding Cauchy problem for the wave equation. This is the classical principle of images.

We shall see that, with the transparent boundary condition (1.2), the answer to our question is still very simple except that we cannot keep exactly the wave equation. Let us make our assertion :

THEOREM 1

Let $\tilde{u} \in C^1(\mathbb{R}^+; L^2(\mathbb{R})) \cap C^0(\mathbb{R}^+; L^2(\mathbb{R}))$ *the unique weak solution of the following problem :*

(1.7) $$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} + 2\,\delta(x)\frac{\partial \tilde{u}}{\partial t} = 0$$

(1.8) $$\tilde{u}(x,0) = \tilde{u}_0(x) \qquad \frac{\partial \tilde{u}}{\partial t}(x,0) = \tilde{u}_1(x)$$

where $\delta(x)$ *is the Dirac distribution and where* $\tilde{u}_0$ *and* $\tilde{u}_1$ *are deduced from* u_0 *and* u_1 *by symmetry :*

(1.9) $$\begin{cases} \tilde{u}_j(x) = u_j(x) & \textit{if} \quad x < 0 \quad j = 0,1 \\ \tilde{u}_j(x) = u_j(-x) & \textit{if} \quad x > 0 \quad j = 0,1 \end{cases}$$

then the restriction u *of* $\tilde{u}$ *to the half line* $x < 0$ *coincides with the solution of* ((1.1),(1.2),(1.3)).

Proof

It is simple and we omit the details :

(i) Let us first precise what we understand of weak solution of ((1.7),(1.8)). By definition, it is a function $\tilde{u}$ in $C^0(\mathbb{R}^+; H^1(\mathbb{R})) \cap C^1(\mathbb{R}^+; L^2(\mathbb{R}))$ such that :

(1.10) $$\begin{cases} \forall\, v \in H^1(R) \quad \dfrac{d^2}{dt^2}\,(\tilde{u}(t), v) + \dfrac{d}{dt}\, b(\tilde{u}(t), v) + a(\tilde{u}(t), v) = 0 \\ \tilde{u}(0) = u_0 \qquad \dfrac{d\tilde{u}}{dt}\,(0) = u_1 \end{cases}$$

where we have set, for $(u,v) \in H^1(\mathbb{R})^2$:

$$(1.11) \qquad \begin{cases} (u,v) = \displaystyle\int_{\mathbb{R}} uv\,dx \\ a(u,v) = \displaystyle\int_{\mathbb{R}} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx \\ b(u,v) = 2\,u(0)\,v(0) \end{cases}$$

It is easy to derive a formal a priori estimate for u by multiplying equation (1.7) by $\frac{\partial u}{\partial t}$ and integrating over $\mathbb{R}$ which leads to :

$$(1.12) \qquad \frac{1}{2}\frac{d}{dt}\left\{\int_{\mathbb{R}} \left|\frac{\partial u}{\partial t}\right|^2 dx + \int_{\mathbb{R}} \left|\frac{\partial u}{\partial x}\right|^2 dx\right\} + 2\left|\frac{\partial u}{\partial t}(0,t)\right|^2 = 0$$

Such an identity can be justified by the use of the Galerkin's method (see [9]) which leads to a uniqueness and existence result.

(ii) Let us assume that :

$$(1.13) \qquad (u_0,u_1) \in H_0^2(\mathbb{R}^-) \times H_0^1(\mathbb{R}^-)$$

Thus $(\tilde{u}_0,\tilde{u}_1)$ belongs to the space $H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and it is easy to derive the following regularity result for the solution $\tilde{u}$:

$$\begin{cases} \tilde{u} \in C^0(\mathbb{R}\,;H^2(\mathbb{R}^+ \cup \mathbb{R}^-)) \\ \tilde{u} \in C^1(\mathbb{R}\,;H^1(\mathbb{R})) \end{cases}$$

A priori, at each time t, the solution $u(t)$ does not belong to the space $H^2(\mathbb{R})$. In fact, one can see that $\frac{\partial \tilde{u}}{\partial x}$ is discontinuous at $x = 0$. Indeed, one has at each time :

$$(1.14) \qquad \frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} = 0 \qquad \text{if } x < 0 \text{ or } x > 0$$

Using a regular test function v we have :

(1.15)
$$\left|\begin{array}{l} \dfrac{d^2}{dt^2}\left(\displaystyle\int_{\mathbb{R}^+} \tilde{u}(t)\, v\, dx\right) - \displaystyle\int_{\mathbb{R}^+} \dfrac{\partial^2 \tilde{u}}{\partial x^2} v\, dx = 0 \\ \dfrac{d^2}{dt^2}\left(\displaystyle\int_{\mathbb{R}^-} \tilde{u}(t)\, v\, dx\right) - \displaystyle\int_{\mathbb{R}^-} \dfrac{\partial^2 \tilde{u}}{\partial x^2} v\, dx = 0 \end{array}\right.$$

By an integration by parts we have :

(1.16)
$$\left|\begin{array}{l} -\displaystyle\int_{\mathbb{R}^+} \dfrac{\partial^2 \tilde{u}}{\partial x^2}(t)\, v\, dx = \displaystyle\int_{\mathbb{R}^+} \dfrac{\partial \tilde{u}}{\partial x}(t) \dfrac{\partial v}{\partial x}\, dx + \dfrac{\partial \tilde{u}}{\partial x}(0^+,t)\, v(0) \\ -\displaystyle\int_{\mathbb{R}^-} \dfrac{\partial^2 \tilde{u}}{\partial x^2}(t)\, v\, dx = \displaystyle\int_{\mathbb{R}^-} \dfrac{\partial \tilde{u}}{\partial x}(t) \dfrac{\partial v}{\partial x}\, dx + \dfrac{\partial \tilde{u}}{\partial x}(0^-,t)\, v(0) \end{array}\right.$$

Plugging (1.16) in (1.15), we obtain, after summation

(1.17)
$$\frac{d^2}{dt^2}(\tilde{u}(t), v) + a(\tilde{u}(t), v) + \left(\frac{\partial \tilde{u}}{\partial x}(0^+,t) - \frac{\partial \tilde{u}}{\partial x}(0^-,t)\right) v(0) = 0$$

Identifying this equality with (1.10) we deduce that :

(1.18)
$$\frac{\partial \tilde{u}}{\partial x}(0^+,t) - \frac{\partial \tilde{u}}{\partial x}(0^-,t) = 2\frac{\partial \tilde{u}}{\partial t}(0,t)$$

(iii) As the initial data $\tilde{u}_0(x)$ and $\tilde{u}_1(x)$ are by construction even functions, it is clear that $\tilde{u}(x,t)$ is even with respect to x. Indeed it is easy to see that the function :

$$\overline{u}(x,t) = \tilde{u}(-x,t)$$

is another weak solution of ((1.7),(1.8)). It is then sufficient to use the uniqueness result to conclude. As $\tilde{u}(x,t) = -\tilde{u}(-x,t)$, we easily deduce that :

(1.19)
$$\frac{\partial \tilde{u}}{\partial x}(0^-,t) = -\frac{\partial \tilde{u}}{\partial x}(0^+,t)$$

With (1.18), we deduce that

(1.20)
$$\frac{\partial \tilde{u}}{\partial x}(0^+,t) = -\frac{\partial \tilde{u}}{\partial x}(0^-,t) = -\frac{\partial \tilde{u}}{\partial t}(0,t)$$

Now let us consider $u = \tilde{u}|_{\mathbb{R}^-}$. We have proved that :

$$(1.21) \qquad \left| \begin{array}{ll} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 & x < 0\,,\, t > 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0 & x = 0\,,\, t > 0 \\ \left\{ \begin{array}{l} u(x,0) = u_0(x) \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \end{array} \right. & \begin{array}{l} x < 0 \\ x < 0 \end{array} \end{array} \right.$$

which shows that u is the unique solution of our model problem (1.1). The theorem is thus proved under the regularity assumption (1.13). In fact it is easy to extend the proof to the general case corresponding to assumptions (1.5) by means of density and continuity arguments. ■

Theorem 1.1 gives us an "algorithm" to solve our model problem ((1.1),(1.2),(1.3)) which consists of :

- extending the data by symmetry
- solve the corresponding Cauchy problem on the line associated with equation (1.7)
- take the restriction to the half line $x < 0$.

Before going to the generalization to higher space dimensions, let us make some comments :

- our image principle is of course extendable to the case of variable coefficients. It suffices to extend them by evenness, as we did for the initial data. The same observation holds in the case of the presence of a right hand side member in equation (1.1) ;

- by symmetry, it is clear that the restriction of $\tilde{u}$ to the half line $\mathbb{R}^+ = \{x \mid x > 0\}$ the solution of the wave equation in $\mathbb{R}^+$ with the transparent boundary condition $\dfrac{\partial u}{\partial t} - \dfrac{\partial u}{\partial x} = 0$ at $x = 0$;

- equation (1.7) appears as a damped wave equation with a punctual damping term at $x = 0$. This observation permits us to make a link between the transparent boundary condition and an absorbing layer approach (see [1],[4] for instance). Indeed, consider a smooth approximation of the Dirac distribution $\delta(x)$ by $\delta_\varepsilon(x)$ such that :

$$(1.22)\qquad \left|\begin{array}{l} \bullet\ \delta_\varepsilon(x) \in C_0^\infty(R) \quad , \quad \delta_\varepsilon(x) \geq 0 \\ \bullet\ \text{supp}\ \delta_\varepsilon \subset [-\varepsilon,\varepsilon] \\ \bullet\ \displaystyle\int_{\mathbb{R}} \delta_\varepsilon(x)\,dx = 1 \end{array}\right.$$

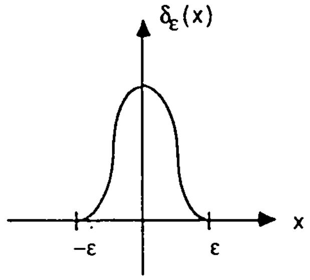


Figure 1.1 : Graph of the function $\delta_\varepsilon(x)$

The solution $\tilde{u}$ of (1.7) can be approximated by the solution $\tilde{u}^\varepsilon$ of the damped wave equation

$$(1.23)\qquad \frac{\partial^2 \tilde{u}^\varepsilon}{\partial t^2} - \Delta \tilde{u}^\varepsilon + 2\,\delta_\varepsilon(x)\frac{\partial \tilde{u}^\varepsilon}{\partial t} = 0$$

(we keep the same initial data as for $\tilde{u}$). By restriction to the half line $x < 0$, it is clear that u, solution of ((1.1),(1.2),(1.3)) can be approximated by $u^\varepsilon = \tilde{u}^\varepsilon|_{\mathbb{R}^-}$ which is characterized by the following set of equations :

$$(1.24)\qquad \left|\begin{array}{ll} \dfrac{\partial^2 u^\varepsilon}{\partial t^2} - \Delta u^\varepsilon + 2\,\delta_\varepsilon(x)\dfrac{\partial u^\varepsilon}{\partial t} = 0 & x < 0\,,\ t > 0 \\ \dfrac{\partial u^\varepsilon}{\partial x} = 0 & \\ u^\varepsilon(x,0) = u_0(x) & \\ \dfrac{\partial u^\varepsilon}{\partial t}(x,0) = u_1(x) & \end{array}\right.$$

Thus u naturally appears as the limit of the solution of the wave equation on the half line $\mathbb{R}^-$, with an absorbing layer of thickness ε in the interval $[-\varepsilon,0]$ and a Neumann boundary condition at $x = 0$ (see fig.1.2)

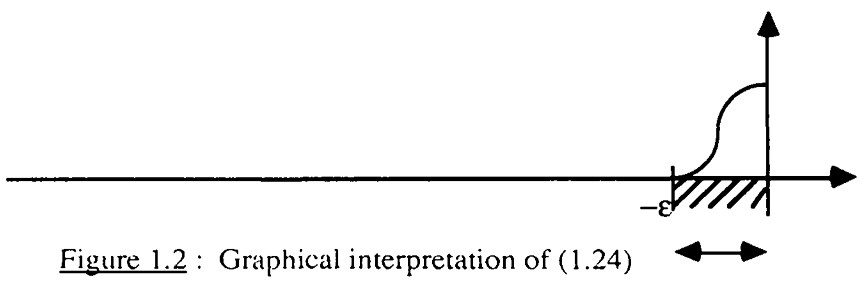


Figure 1.2 : Graphical interpretation of (1.24)

• because of the link between equation (1.7) and transparent boundary conditions in both domains $\mathbb{R}^-$ and $\mathbb{R}^+$, it is clear that the damping term at $x = 0$ plays the role of a "black hole" which absorbs all the energy of the solution. For instance if the initial data of problem ((1.7),(1.8)) are even and compactly supported, the solution is equal to 0 in a finite time.

2. A FORMAL GENERALIZATION TO THE HIGHER DIMENSIONAL CASE

We consider the wave equation in a half-space of dimension d

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \qquad \text{for } (x,t) \in \mathbb{R}^d_- \times \mathbb{R}_+ \tag{2.1}$$

We shall denote the spatial variable X by

$$X = (x,y) \qquad \text{with} \quad x < 0 \,,\, y \in \mathbb{R}^{d'} \quad d' = d-1 \,,$$

and the boundary hyperplane of $\mathbb{R}^d_-$ by

$$\Gamma = \{(0,y)\,;\, y \in \mathbb{R}^{d'}\}$$

The trace of u on $\Gamma \times \mathbb{R}_+$ will be written u_Γ or simply u if no confusion can occur. We consider on Γ a general transparent (or absorbing) boundary condition

$$\left(\frac{\partial u}{\partial x}\right)_\Gamma + B(u_\Gamma) = 0 \tag{2.2}$$

where the operator B is given by :

$$\begin{cases} B(u_\Gamma) = \dfrac{\partial u_\Gamma}{\partial t} - \displaystyle\sum_{k=1}^{K} \beta_k \frac{\partial \varphi_k}{\partial t} & \text{(i)} \\ \dfrac{\partial \varphi_k}{\partial t} - \alpha_k \, \Delta_\Gamma \, \varphi_k = \Delta_\Gamma \, u_\Gamma & \text{(ii)} \end{cases} \tag{2.3}$$

with mutually distinct constants α_k, β_k. In (2.3,ii) Δ_Γ denote the tangential laplacian. The condition (2.2) is obtained if one substitutes to the dispersion relation

$$\xi = -\tau\sqrt{1 - |\eta|^2/\tau^2}$$

of the exact transparent condition , an approximate dispersion relation

$$\xi = -\tau\left(1 - \sum_k \frac{\beta_k\left(|\eta|^2/\tau^2\right)}{1 - \alpha_k\left(|\eta|^2/\tau^2\right)}\right)$$

where ξ,η,τ designate the dual (Fourier) variables of x,y,t respectively, and the rational function

$$P(s) = 1 - \sum_k \frac{\beta_k\, s^2}{1 - \alpha_k\, s^2}$$

is a rational approximation of the function $r(s) = \sqrt{1 - s^2}$. In [13], one can find all details about and historic of the boundary conditions of the type (2.2). In particular, if B is given in (2.3), an application of the Theorem 2 of this paper yields the following necessary and sufficient condition for the strong well-posedness of the initial boundary value problem associated to (2.1-2.2) :

$$\text{(2.4)} \qquad \begin{cases} \beta_k > 0 \quad , \quad 0 \le \alpha_k < 1 \, , \\ \displaystyle\sum_k \frac{\beta_k}{1 - \alpha_k} < 1 \end{cases}$$

In the following sections, we shall restrict ourselves to the two cases of the so-called first order (K = 0, no necessity of φ_k) and second order (K = 1 , $\alpha_1 = 0$) absorbing boundary conditions [6]. But now, let us show, for all conditions of the type (2.2), how to generalize the principle of images to the associated initial boundary value problem (IBVP).

<u>*THEOREM 2*</u>

a) Let u *be a classical solution of the IBVP* (2.1, 2.2 *and* 2.5), *where*

$$\text{(2.5)} \qquad \begin{cases} u(X,0) = u_0(X) \\ \dfrac{\partial u}{\partial t}(X,0) = u_1(X) \end{cases}$$

and let $\tilde{u}, \tilde{f}, \tilde{u}_0, \tilde{u}_1$ *be the extensions of* u, f, u_0, u_1 *over* $\mathbb{R}^d_+$ *by symmetry, then* $\tilde{u}$ *is solution of the following initial value problem*

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \Delta \tilde{u} + 2\, \delta_\Gamma \otimes B(\tilde{u}_\Gamma) = \tilde{f} \tag{2.6}$$

$$\begin{cases} \tilde{u}(X,0) = \tilde{u}_0(X) \\ \dfrac{\partial \tilde{u}}{\partial t}(X,0) = \tilde{u}_1(X) \end{cases} \tag{2.7}$$

where δ_Γ *is the Dirac distribution on* Γ.

b) Conversely, if the problem (2.6-2.7) *has a unique solution for* $\tilde{f}, \tilde{u}_0, \tilde{u}_1$, *then its restriction to the half space* $\mathbb{R}^d_-$ *is a solution of the IBVP* (2.1, 2.2, 2.5).

Proof

It is a simple application of the well known jump formula (cf. e.g. [ ]). If $v \in \mathcal{C}^2(\mathbb{R}^d_-) \cap \mathcal{C}^1(\overline{\mathbb{R}^d_-})$, or more generally if $v \in \mathcal{D}'(\mathbb{R}^d_-)$ and $(v, \frac{\partial v}{\partial x}) \in \mathcal{C}(\overline{\mathbb{R}}_-, \mathcal{D}'(\mathbb{R}^{d'}))$, then the extension by 0 of v over $\mathbb{R}^d_+$ is a distribution v_0 of $\mathbb{R}^d$ satisfying the formula :

$$\frac{\partial^2 v_0}{\partial x^2} = \left(\frac{\partial^2 v}{\partial x^2}\right)_0 - \left(\delta_\Gamma \otimes \frac{\partial v}{\partial x}(0,y) + \frac{\partial}{\partial x} \delta_\Gamma \otimes v(0,y)\right) \tag{2.8}$$

On the other side, there is no jump for the tangential derivatives of v_0 so that :

$$\Delta v_0 = (\Delta v)_0 - \left(\delta_\Gamma \otimes \frac{\partial v}{\partial x}(0,y) + \frac{\partial \delta_\Gamma}{\partial x} \otimes v(0,y)\right) \tag{2.9}$$

Now, if $v \in \mathcal{D}'(\mathbb{R}^d)$ and $v \in \mathcal{C}(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{d'}))$, $\frac{\partial v}{\partial x} \in \mathcal{C}(\overline{\mathbb{R}}_+, \mathcal{D}'(\mathbb{R}^{d'})) \cap \mathcal{C}(\overline{\mathbb{R}}_-, \mathcal{D}'(\mathbb{R}^{d'}))$, we have

$$\Delta v = (\Delta v)_F - \delta_\Gamma \otimes \left[\frac{\partial v}{\partial x}\right](y) \tag{2.10}$$

where $[g] = g(0_-,y) - g(0_+,y)$ is the jump of g over Γ, and $(\Delta v)_F$ is the sum of $\Delta(v|_{\mathbb{R}^d_-})$ extended by 0 in $\mathbb{R}^d_+$ and $\Delta(v|_{\mathbb{R}^d_+})$ extended by 0 in $\mathbb{R}^d_-$. This is the "function part" of Δv, because if Δv is equal to a locally summable function w on $\mathbb{R}^d_+ \cup \mathbb{R}^d_-$, then

$$\langle (\Delta v)_F, \phi \rangle = \int_{\mathbb{R}^d_+ \cup \mathbb{R}^d_-} w(X)\, \phi(X)\, dX \qquad \forall\, \phi \in \mathcal{D}(\mathbb{R}^d).$$

Applying (2.10) to the function $\tilde{u}$, which is even in the variable x, one gets (2.6). The other part of the theorem is obvious. ■

Now, our "generalized principle of images", that is, the equivalence of (2.1, 2.2 and 2.5) and (2.6, 2.7 with even data $\tilde{f}$, $\tilde{u}_0$, $\tilde{u}_1$), will be entirely proved if a uniqueness result for the last problem can be obtained. For the first order absorbing boundary condition, this can be made with an analogous procedure as in section 1. In section 3, we propose however another approach, valid also for the second order condition. For later referencing, we shall call the equation (2.6) the "equation of images".

3. A MATHEMATICAL ANALYSIS OF THE EQUATION OF IMAGES FOR THE FIRST ORDER ABSORBING BOUNDARY CONDITION

In this case, our equation of images can be written compactly :

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + 2\,\delta_\Gamma \otimes \frac{\partial u_\Gamma}{\partial t} = f \tag{3.1}$$

We have dropped the tilde sign in (2.6), and we shall not suppose the evenness with respect to x of neither the source function f nor the Cauchy data u_0, u_1.

We propose in this section an adaptation of the Hille-Yosida theory (cf. e.g. [11]) to solve the Cauchy problem of (3.1). For that, we consider u as a function of t, with values in some spaces of distributions on $\mathbb{R}^d$. Denoting by v the time derivative $\frac{\partial u}{\partial t}$ of u, and by U(t) the vectorial function $(u(t)\,,\,v(t))^T$, we can write (3.1) as an evolution equation of first order :

$$\frac{dU}{dt} + AU = F \tag{3.2}$$

where $F = (0,f)^T$ and A the matrix operator defined by

$$A = \begin{pmatrix} 0 & -I \\ -\Delta & 2\,\delta_\Gamma \otimes \gamma \end{pmatrix} \tag{3.3}$$

γ designates the trace operator defined for sufficiently regular distributions on $\Omega = \mathbb{R}^d_+ \cup \mathbb{R}^d_-$. More precisely, we denote by $\gamma_+ u$ (resp. $\gamma_- u$) the limit (when it exists) of u(x,y) when x tends to 0 with positive (resp. negative) values, and we shall use the notation γu when $\gamma_+ u = \gamma_- u$.

We consider A in (3.3) as a unbounded operator on the Sobolev space

$$\mathcal{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$$

with domain

$$\text{(3.4)} \qquad \begin{aligned} \mathcal{D}(A) = \Big\{ U = (u,v)^T \; ; \; & u \in H^2(\Omega) \cap H^1(\mathbb{R}^d) ; \\ & v \in H^1(\mathbb{R}^d) \qquad ; \\ \text{such that} \quad & \left[\frac{\partial u}{\partial x}\right] + 2\gamma v = 0 \Big\} \end{aligned}$$

Our main result is :

THEOREM 3
The operator $-\left(A + \frac{1}{2} I\right)$ *is the infinitesimal generator of a semi-group of contractions of class* C^0 *on* $\mathcal{H}$.

Proof
We divide the proof into 3 parts :

a) A maps $\mathcal{D}(A)$ into $\mathcal{H}$
From $u \in H^2(\Omega) \cap H^1(\mathbb{R}^d)$ and $v \in H^1(R^d)$, the condition (3.4) has a precise sense in $H^{1/2}(\Gamma)$. On the other side, the jump formula (2.10) is valid for u, with $(\Delta u)_F \in L^2(\mathbb{R}^d)$. Then

$$\text{(3.5)} \qquad AU = \begin{pmatrix} -v \\ -(\Delta u)_F + \delta_\Gamma \otimes \left(\left[\frac{\partial u}{\partial x}\right] + 2\gamma v \right) \end{pmatrix} = \begin{pmatrix} -v \\ -(\Delta u)_F \end{pmatrix} \in \mathcal{H}$$

b) The operator $A + \frac{I}{2}$ is monotone
For $U \in \mathcal{D}(A)$, by (3.5) and the formula of the integrations by parts :

$$(AU,U)_{\mathcal{H}} = \int_{\mathbb{R}^d} \left(- u\bar{v} - \nabla u.\nabla \bar{v}\right) dx + \int_{\mathbb{R}^d} \left(- (\Delta u)_F .\bar{v}\right) dx$$
$$= \int_{\mathbb{R}^d} (- u\bar{v}\,) \, dx - \int_{\Gamma} \left[\frac{\partial u}{\partial x}\right] \bar{v}\, dy$$

Then, by (3.4)

$$(AU,U)_{\mathcal{H}} = - \int_{\mathbb{R}^d} u\,\bar{v}\, dx + 2 \int_{\mathbb{R}^{d-1}} |v(0,y)|^2\, dy$$

while

$$(U,U)_{\mathcal{H}} = - \int_{\mathbb{R}^d} \left[\left(|u|^2 + |\nabla u|^2\right) + |v|^2\right] dx$$

The conclusion follows immediately. One has even the more precise inequality :

(3.6) $$((A+\mu I)\, U,U)_{\mathcal{H}} \geq 0 \qquad \forall\, U \in \mathcal{D}(A) \text{ and } \forall\, \mu \geq \frac{1}{2}\,.$$

c) <u>For $\lambda > 0$, the range $R(\lambda I+A)$ of $\lambda I+A$ is $\mathcal{H}$</u>

We have to prove that for all $(f_1,f_2) \in H^1 \times L^2$ (we drop the domain of the Sobolev space when it is the whole space $\mathbb{R}^d$), there exists $U \in \mathcal{D}(A)$ such that

(3.7) $$\begin{cases} \lambda u - v = f_1 \\ \lambda v - (\Delta u)_F = f_2\,. \end{cases}$$

With the condition between u and v in (3.4), and the jump formula (2.10), one can eliminate v in (3.7) to obtain the following problem for u:

(3.8) $$-\Delta u + \lambda^2 u + 2\,\lambda\delta_\Gamma \otimes (\gamma u) = \lambda f_1 + f_2 + 2\delta_\Gamma \otimes f_1$$

The variational formulation of (3.8) is:

$$(3.9)\quad \begin{cases} \text{To find } u \in H^1 \text{ such that:} \\ \displaystyle\int_{\mathbb{R}^d} (\nabla u.\nabla w + \lambda^2 uw)dX + 2\lambda \int_\Gamma (\gamma u)(\gamma w)dy \\ \displaystyle = \int_{\mathbb{R}^d} (\lambda f_1 + f_2)w dX + 2\int_\Gamma (\gamma f_1)(\gamma w)dy \qquad \forall w \in H^1 \end{cases}$$

It is clear that, as soon as $\lambda>0$, one can apply Lax-Milgram's theorem to solve (3.9). Now, setting $v=\lambda u+f_1$ and reinterpreting (3.9) in the sense of distributions, one easily checks that $U=(u,v)$ satisfies (3.7) and belongs to D(A). Part c/ is proved.

The conclusion of the theorem follows then the (direct part of the) Lumer-Philipps theorem.[11] ■

Coming back to the initial boundary value problem of the wave equation with the first order absorbing boundary condition, and its equivalent equation of images, we have :

<u>*THEOREM 4*</u>

a) For $f \in \mathcal{C}^1([0,T];L^2)$, $u_0 \in H^2(\Omega) \cap H^1$ *and* $u_1 \in H^1$ *satisfying the condition*

$$(3.10)\qquad \left[\frac{\partial u_0}{\partial x}\right] + 2\,\gamma u_1 = 0 \quad \textit{on } \Gamma$$

The Cauchy problem for equation (3.1) *with initial data* (u_0,u_1) *admits a unique solution with*

$$(3.11)\qquad u \in \mathcal{C}([0,T];H^2(\Omega)\cap H^1)\ ;\ \frac{du}{dt} \in \mathcal{C}([0,T];H^1)\ ;\ \frac{d^2u}{dt^2} \in \mathcal{C}([0,T];L^2)$$

Moreover, if $f=0$, *the solution* u *satisfies the following energy identity :*

$$(3.12)\qquad \frac{d}{dt}\left(\frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right)dx\right) + 2\int_\Gamma \left|\frac{\partial u}{\partial t}\right|^2 dy = 0$$

b) For $f \in \mathcal{C}^1([0,T];L^2(\mathbb{R}^d))$, $u_0 \in H^2(\mathbb{R}^d)$ *and* $u_1 \in H^1(\mathbb{R}^d)$ *such that the following condition is satisfied :*

$$(3.13)\qquad \frac{\partial u_0}{\partial x} + \gamma u_1 = 0 \quad \textit{on } \Gamma$$

The IBVP (2.1-2.2-2.3 *with* $K=0$) *is well-posed and its solution equal to the restriction of the*

Cauchy problem (3.1) *with* f, u_0, u_1 *extended by symmetry to* $\mathbb{R}^d_+$.
Moreover, if f = 0, *one has the following energy identity :*

(3.14) $$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^d_-}\left(|\nabla u|^2+\left|\frac{\partial u}{\partial t}\right|^2\right)dx\right)+\int_\Gamma\left|\frac{\partial u}{\partial t}\right|^2 dy=0$$

Proof

The well-posedness parts of the theorem are a mere translation of the general theory of linear evolution theory to our case.

The energy identities (3.12) and (3.14) are obtained by the classical technique.` ■

Remarks

(i) The condition (3.10) and (3.13) are obviously satisfied if $u_0 \in H^2_0(\Omega)$, $u_1 \in H^2_0(\Omega)$ (resp. $u_0 \in H^2_0(\mathbb{R}^d_-)$ and $u_1 \in H^1_0(\mathbb{R}^d_-)$, or more simply, if the supports of u_0 and u_1 do not touch Γ. That is the case in problems where the boundary Γ is artificial and can be placed as we like.

(ii) When the initial data (u_0,u_1) belongs to $\mathcal{H}$ only, not to $\mathcal{D}(A)$, one can consider $U(t) = T(t)U_0$, where $T(t)$ is the semi-group of operators associated to $(-A)$, as a weak solution of the Cauchy problem for (3.2). That yields a weak solution of (3.1) with instead of (3.11) :

(3.15) $$u \in \mathcal{C}([0,T];H^1)\ ;\ \frac{\partial u}{\partial t} \in \mathcal{C}([0,T];L^2)$$

In this case, the energy identity (3.12) cannot be written as such, however by standard density arguments, it is easy to show that the energy of u is still well defined and remains a decrease function of t.

4. THE CASE OF THE SECOND ORDER ABSORBING BOUNDARY CONDITION

We study in this section the "second order equation of images" :

(4.1) $$\frac{\partial^2 u}{\partial t^2} - \Delta u + 2\,\delta_\Gamma \otimes B(u_\Gamma) = f$$

where $u_\Gamma = \gamma u$ and

(4.2)
$$\begin{cases} B(u_\Gamma) = \dfrac{\partial u_\Gamma}{\partial t} - \beta\varphi \\ \dfrac{\partial \varphi}{\partial t} = \Delta_\Gamma(\gamma u) \end{cases}$$

Denoting by $v = \dfrac{\partial u}{\partial t}$, we can write (4.1) as an evolution equation like (3.2) but with now $U = (u,v,\varphi)^T$ and

(4.3)
$$A = \begin{pmatrix} 0 & -I & 0 \\ -\Delta & 2\,\delta_\Gamma \otimes \gamma & -2\beta\delta_\Gamma \\ -\Delta_\Gamma \circ \gamma & 0 & 0 \end{pmatrix}$$

For reasons which will appear further, we shall consider A as a unbounded operator on the following space

$$\mathcal{H} = \{U=(u,v,\varphi) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \text{ such that (4.4) holds}\}$$

where

$$\mathcal{H}_1 = H^2(\Omega) \cap H^1$$
$$\mathcal{H}_2 = H^1$$
$$\mathcal{H}_3 = H^{1/2}(\Gamma)$$

and

(4.4)
$$\left[\frac{\partial u}{\partial x}\right] + 2\,(\gamma v - \beta\varphi) = 0$$

with the domain defined by

$$\mathcal{D}(A) = \{U=(u,v,\varphi) \in \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3 \text{ such that (4.4) and (4.5) hold}\}$$

where

$$\mathcal{D}_1 = \{u \in H^3(\Omega) \cap H^1 \; ; \; \gamma_+(\Delta u) = \gamma_-(\Delta u)\}$$
$$\mathcal{D}_2 = \mathcal{H}_1$$
$$\mathcal{D}_3 = H^{3/2}(\Gamma)$$

and

(4.5)
$$\left[\frac{\partial v}{\partial x}\right] + 2\,(\gamma(\Delta u) - \beta\,\Delta_\Gamma(\gamma u)) = 0$$

Thus, the boundary condition for u is now put in the space $\mathcal{H}$ while in $\mathcal{D}(A)$ we have taken into account both the boundary conditions for u and v.

However, these (usual) tricks are not sufficient for our purpose, and later calculations lead us to define on $\mathcal{H}_1$ and $\mathcal{H}_2$ other norms than the usual ones, to control the behavior of the 'normal derivatives' $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ in the interior of the domain, according to the following lemma:

LEMMA 1

For $\theta > 0$, the following norms are equivalent to the classical ones in the concerned spaces :

(4.6) $$\|u\|_{\mathcal{H}_1} = \left[\int_\Omega \left(|u|^2 + |\nabla u|^2 + |\Delta u|^2 + \theta \left|\nabla\left(\frac{\partial u}{\partial x}\right)\right|^2\right) dX\right]^{1/2}$$

(4.7) $$\|v\|_{\mathcal{H}_2} = \left[\int_\Omega \left(|v|^2 + |\nabla v|^2 + \theta \left|\left(\frac{\partial v}{\partial x}\right)\right|^2\right) dX\right]^{1/2}$$

Proof

The $\mathcal{H}_2$ part is obvious. For $\mathcal{H}_1$, we note first that $H^2(\Omega) \cap H^1$ is a closed subspace of $H^2(\Omega)$ defined by

$$H^2(\Omega) \cap H^1 = \{u \in H^2(\Omega) \ ; \ \gamma_+ u = \gamma_- u\}$$

and its natural norm is then the one inheritated from $H^2(\Omega)$:

$$\|u\|_{H^2(\Omega) \cap H^1} = \left[\sum_{|\alpha| \leq 2} \int |D^\alpha u|^2 \, dx\right]^{1/2}$$

which is clearly greater than the second member of (4.6).

`For the inverse inequality, we can use the following lemma :

LEMMA 2

The space

$$V = \left\{u \in H^1(\mathbb{R}^d_+) \, ; \, \Delta u \in L^2(\mathbb{R}^d_+) \, , \, \frac{\partial^2 u}{\partial x^2} \in L^2(\mathbb{R}^d_+)\right\}$$

is algebraically and topologically identical to $H^2(\mathbb{R}^d_+)$.

Proof

Let $\tilde{u} = pu$ where p is the extension operator by reflexions from $\mathbb{R}^d_+$ to $\mathbb{R}^d$ (cf. [9]) :

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & x > 0 \\ \sum_1^2 \alpha_j\, u(-jx,y) & x < 0 \end{cases}$$

where the α_j are defined by

$$\sum_{j=1}^{2} (-1)^i j^i \alpha_j = 1 \qquad i = 0,1 .$$

Thus $\alpha_1 = 3, \alpha_2 = -2$.
It is easy to verify that $\tilde{u} \in H^1(\mathbb{R}^d)$ and $\Delta\tilde{u} \in L^2(\mathbb{R}^d)$. A Fourier calculus proves then $\tilde{u} \in H^2(\mathbb{R}^d)$, so that $u = \tilde{u}|_{\mathbb{R}^d_+}$ belongs to $H^2(\mathbb{R}^d_+)$.
Moreover, for $x < 0$,

$$\Delta\tilde{u}(x,y) = \alpha_1\, \Delta u(-x,y) + \alpha_2 \sum_{2 \le k \le d} \frac{\partial^2 u}{\partial y_k^2}(-2x,y) + 4\, \alpha_2 \frac{\partial^2 u}{\partial x^2}(-2x,y)$$

The hypothesis concerning $\dfrac{\partial^2 u}{\partial x^2}$ in V permits to conclude the lemma. ■

Finally, it is clear that $\mathcal{H}$ is a closed subspace of $\mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$, isomorphic to $\mathcal{H}_1 \times \mathcal{H}_2$, and we can equip it with the norm of $\mathcal{H}_1 \times \mathcal{H}_2$:

(4.8) $$\|U\|_{\mathcal{H}} = \left(\|u\|^2_{\mathcal{H}_1} + \|v\|^2_{\mathcal{H}_2}\right)^{1/2} ,$$

which shows up the auxiliary character of the function φ.

The reason of our choice of the norms (4.6, 4.7) appears now in the following

LEMMA 3

If $0 < \beta < 1$, *and* $\theta = \dfrac{\beta}{1-\beta}$, *the operator* $I+A$ *is monotone on* $\mathcal{H}$.

Proof

Let $U \in \mathcal{D}(A)$. By the condition (4.4) and the jump formula (2.10), we have

$$AU = \begin{pmatrix} -v \\ -(\Delta u)_F \\ -\Delta_\Gamma(\gamma u) \end{pmatrix}$$

It is easy then to verify that $AU \in \mathcal{H}$.

Now, by (4.8)

$$(AU,U)_{\mathcal{H}} = (-v,u)_{\mathcal{H}_1} + \left(-(\Delta u)_F,v\right)_{\mathcal{H}_2}$$

where, conforming to (4.6):

$$(-v,u)_{\mathcal{H}_1} = -\int_?^? (u\bar{v}+\nabla u.\nabla\bar{v} + \Delta u.\Delta\bar{v} + \theta\nabla(\frac{\partial u}{\partial x}).\nabla(\frac{\partial v}{\partial x}))dX$$

For $-(\Delta u)_F,v)_{\mathcal{H}_2}$, by integrations by parts, we obtain:

$$\begin{aligned}\left(-(\Delta u)_F,v\right)_{\mathcal{H}_2} &= \int_\Omega \nabla u.\nabla\bar{v}\,dx - \int_\Gamma \left[\frac{\partial u}{\partial x}\right]\bar{v}\,dy \\ &+ \int_\Omega \Delta u.\Delta\bar{v}\,dx - \int_\Gamma \left[\frac{\partial \bar{v}}{\partial x}\right]\gamma\left(\Delta u\right)dy \\ &+ \theta\int_\Omega \nabla\left(\frac{\partial u}{\partial x}\right).\nabla\left(\frac{\partial \bar{v}}{\partial x}\right)dx - \theta\int_\Gamma \left[\frac{\partial \bar{v}}{\partial x}\right]\gamma\left(\frac{\partial^2 u}{\partial x^2}\right)dy\end{aligned}$$

Taking into account the conditions (4.5), and $\theta = \dfrac{\beta}{1-\beta}$, one gets :

$$(AU,U)_{\mathcal{H}} = -\int_\Omega u\,\bar{v}\,dx - \int_\Gamma \left[\frac{\partial u}{\partial x}\right]\bar{v}\,dy + \frac{1}{2(1-\beta)}\int_\Gamma \left\|\left[\frac{\partial v}{\partial x}\right]\right\|^2 dy \tag{4.9}$$

The last term in (4.9) is positive, the other terms easily absorbed by the terms in $(U,U)_{\mathcal{H}}$. Thus the lemma is proved. ■

Now, we can prove for A the same result as in the first order case :

THEOREM 5

For $0 < \beta < 1$, *the operator* $-(A+I)$ *is the infinitesimal generator of a semi-group of contractions of class* C^0 *in* $\mathcal{H}$.

Proof

After lemma 3, what is left to prove is that for some $\lambda > 1$, $\lambda I + A$ is a surjective operator from $\mathcal{D}(A)$ to $\mathcal{H}$. In fact, this is true for all $\lambda > 0$, and the proof follows the same path as in part c) of theorem 3.

We have to solve the system

$$
(4.10) \qquad
\begin{gathered}
U = (u,v,\varphi) \in \mathcal{D}(A) \text{ such that} \\
\begin{cases}
\lambda u - v = f_1 \\
\lambda v - (\Delta u)_F = f_2 \\
\lambda \varphi - \Delta_\Gamma(\gamma u) = f_3
\end{cases}
\end{gathered}
$$

with $F = (f_1,f_2,f_3) \in \mathcal{H}$.

The same approach as in part c) of theorem 3 will give us $u \in H^3(\Omega) \cap H^1$, $v \in H^1$, $\varphi \in H^{3/2}(\Gamma)$ satisfying the condition (4.4), for $F \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$. Moreover, by (4.10,ii), $(\Delta u)_F$ belongs to H^1, so that $\gamma_+(\Delta u) = \gamma_-(\Delta u)$. The condition (4.4) satisfied by F yields then the condition (4.5) for U. ■

Returning to the equation of images (4.1) and the IBVP for the wave equation with the second order absorbing boundary condition, one gets

THEOREM 6

Suppose that : $0 < \beta < 1$. *Then*

a) *For* $f \in \mathcal{C}^1([0,T]\,;\,H_0^1(\Omega))$, $u_0 \in H_0^3(\Omega)$, $u_1 \in H_0^2(\Omega)$ *and* $\varphi_0 = 0$, *the problem for equation* (4.1) *with initial data* (u_0,u_1,φ_0) *admits a unique solution with*

$$
(4.11) \qquad u \in \mathcal{C}\left([0,T]\,;\,\mathcal{D}_1\right) \cap \mathcal{C}^1\left([0,T]\,;\,\mathcal{H}_1\right) \cap \mathcal{C}^2\left([0,T]\,;\,H_1\right)
$$

and the boundary relation :

$$
(4.12) \qquad \left[\frac{\partial^2 u}{\partial t \partial x}\right] + 2\left(\gamma(\Delta u) - \beta\gamma(\Delta_\Gamma\, u_\Gamma)\right) = 0
$$

Moreover, if $f = 0$, u *satisfies the following energy identity*

Moreover, if $f = 0$, u *satisfies the following energy identity*

$$\frac{d}{dt}\left\{(1-\beta)\int_{\Omega}\left(\left|\frac{\partial^2 u}{\partial t^2}\right|^2+\left|\nabla\frac{\partial u}{\partial t}\right|^2\right)dx+\beta\int_{\Omega}\left(\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2+\left|\nabla\frac{\partial u}{\partial x}\right|^2\right)dx\right\} + \int_{\Gamma}\left\|\left[\frac{\partial^2 u}{\partial x\partial t}\right]\right\|^2 dy = 0 \tag{4.13}$$

b) *For* $f \in \mathcal{C}^1([0,T]\,;H_0^1(\mathbb{R}^d))$, $u_0 \in H_0^3(\mathbb{R}^d)$, $u_1 \in H_0^2(\mathbb{R}^d)$ *and* $\varphi_0 = 0$, *the IBVP for the wave equation with boundary condition defined by* (2.2) *and* (4.2) *is well-posed.Its solution satisfies*

$$u \in \mathcal{C}([0,T]\,;H^3(R^d)) \cap \mathcal{C}^1([0,T]\,;H^2(R^d)) \cap \mathcal{C}^2([0,T]\,;H^1(R^d)) \tag{4.14}$$

and the relation

$$\frac{\partial^2 u}{\partial x\partial t} + \gamma(\Delta u) - \beta\Delta_\Gamma u = 0 \quad on \quad \Gamma \tag{4.15}$$

Moreover, if $f = 0$, u *satisfies the following energy identity :*

$$\frac{d}{dt}\left\{(1-\beta)\,E\left(\frac{\partial u}{\partial t}\right)+\beta\,E\left(\frac{\partial u}{\partial x}\right)\right\}+\int_{\Gamma}\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dy = 0 \tag{4.16}$$

where

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d}\left(|\nabla u|^2+\left|\frac{\partial u}{\partial t}\right|^2\right)dx$$

is the classical energy of u.

<u>Proof</u>

Part b) is a simple application of part a), which is, except for the energy identity (4.13), merely a translation to our problem of the general theory of semi-groups of operators. We note that our assumption for f, u_0, u_1, φ_0 are only a simplified version of the general hypothesis $F \in \mathcal{C}^1([0,T]\,;\mathcal{H})$ and $U_0 \in \mathcal{D}(A)$, with $F = (0,f,0)$ here.

All we have to prove is (4.13) when $f = 0$.

Consider the evolution equation $\frac{dU}{dt} + AU = 0$. Taking the scalar product in $\mathcal{H}$ with U, one gets :

$$\left(\frac{dU}{dt}, U\right)_{\mathcal{H}} + (AU, U)_{\mathcal{H}} = 0$$

Using (4.9), the fact that $\left(\frac{dU}{dt}, U\right)_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathcal{H}}$, and the relation $v = \frac{du}{dt}$, one obtains :

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Bigg\{\iint_{\Omega} \Bigg(|u|^2 + |\nabla u|^2 + \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial^2 u}{\partial t^2}\right|^2 + \left|\nabla \frac{\partial u}{\partial t}\right|^2 \\
+ \frac{\beta}{1-\beta}\left(\left|\frac{\partial^2 u}{\partial x \partial t}\right|^2 + \left|\nabla \frac{\partial u}{\partial x}\right|^2\right)\Bigg) dx\Bigg\} \\
- \int_{\Omega} u \frac{\partial \bar{u}}{\partial t} dx - \int_{\Gamma} \left[\frac{\partial u}{\partial x}\right] \frac{\partial \bar{u}}{\partial t} dy + \frac{1}{2(1-\beta)} \int_{\Gamma} \left\|\left[\frac{\partial^2 u}{\partial x \partial t}\right]\right\|^2 dy = 0
\end{aligned} \tag{4.17}$$

Doing the time derivation for the first three terms in the brackets { }, and using a Green formula, taking into account the relation $\frac{\partial^2 u}{\partial t^2} = \Delta u$, one finds that these terms are compensated exactly by the two terms with the minus sign in (4.17). Consequently, (4.13) is proved.Formula (4.16) is a consequence of (4.13) and theorem 2. ■

Finally, to conclude this section, we note that, like the first order problem, the IBVP for the wave equation with the second order absorbing boundary condition can have a weak solution with decreasing 'second order energy':

$$E_1(u) = \beta\, E\left(\frac{\partial u}{\partial x}\right) + (1-\beta)\, E\left(\frac{\partial u}{\partial t}\right)$$

when weaker assumptions on the data (u_0, u_1, φ_0, f) are made.

This energy estimate does correspond to a strong stability result in the sense of Kreiss (see e.g.[5]) since it enables us to estimate all second order derivatives of the solution (in L^2 norms, at any $t>0$) with the help of the same quantities at $t=0$. The condition $0 < \beta < 1$ corresponds to the particular case of the conditions (2.4). In [6], we obtain similar energy formulas for the general absorbing boundary conditions (2.2,2.3) when the well-posedness conditions (2.4) are satisfied.

5. APPLICATIONS TO STABLE APPROXIMATIONS OF ABSORBING BOUNDARY CONDITIONS COUPLED WITH FOURTH ORDER ACCURATE FINITE DIFFERENCE SCHEMES FOR THE WAVE EQUATION

5.1. The 1-d case.

With our image principle we are now able to give a solution to the problem set up in the introduction, i.e. the construction of discrete absorbing conditions for the 1D wave equation :

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad x < 0 \tag{5.1}$$

when one uses the following fourth order space discretization ($u_j(t)$ approximating $u(x_j,t)$, $x_j = jh$) :

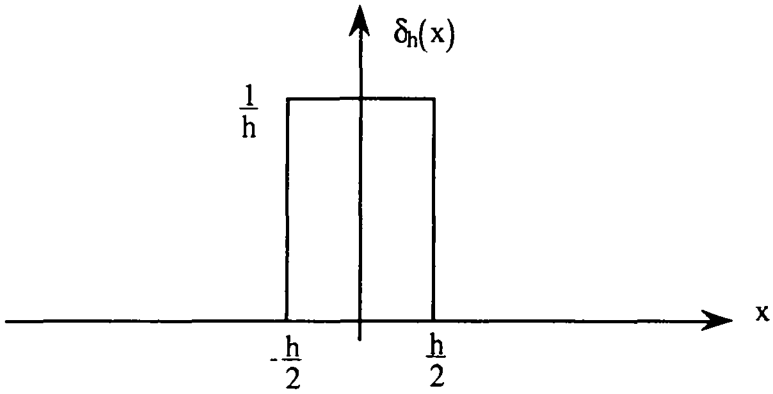
$$\frac{\partial^2 u_j}{\partial t^2} - \frac{4}{3}\,\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \frac{1}{3}\,\frac{u_{j+2} - 2u_j + u_{j-2}}{4h^2} = 0 \tag{5.2}$$

The problem was to find two equations to write for $j = 0$ and $j = -1$ while one only has one boundary condition. With the image principle, this problem disappears since the concept of interior point has no sense any longer : any node can be seen as an interior node ! We have simply to give a discretization for the "extended" problem

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} + 2\,\delta(x)\frac{\partial \tilde{u}}{\partial t} = 0 \tag{5.3}$$

This implies some approximation of the δ-function. The simplest one corresponds to the following choice :

$$\delta(x) \simeq \delta_h(x) \tag{5.4}$$



This leads to the following set of the equations :

$$(5.5)\qquad \begin{cases} \dfrac{d^2\tilde{u}_j}{dt^2} - \dfrac{4}{3}\,\dfrac{\tilde{u}_{j+1} - 2\tilde{u}_j + \tilde{u}_{j-1}}{h^2} + \dfrac{1}{3}\,\dfrac{\tilde{u}_{j+2} - 2\tilde{u}_j + \tilde{u}_{j-2}}{4h^2} = 0 & j \neq 0 \\ \dfrac{d^2\tilde{u}_0}{dt^2} - \dfrac{4}{3}\,\dfrac{\tilde{u}_1 - 2\tilde{u}_0 + \tilde{u}_{-1}}{h^2} + \dfrac{1}{3}\,\dfrac{\tilde{u}_2 - 2\tilde{u}_0 + \tilde{u}_{-2}}{4h^2} = 0 & j = 0 \end{cases}$$

Note that the discrete variational formulation of this problem can be written :

$$(5.6)\qquad \left|\begin{array}{l} \dfrac{d^2}{dt^2}\left(\displaystyle\sum_j \tilde{u}_j\,\tilde{v}_j\,h\right) + \dfrac{4}{3}\displaystyle\sum_j \dfrac{\tilde{u}_{j+1} - \tilde{u}_j}{h}\,\dfrac{\tilde{v}_{j+1} - \tilde{v}_j}{h}\,h - \dfrac{1}{3}\displaystyle\sum_j \dfrac{\tilde{u}_{j+1} - \tilde{u}_{j-1}}{2h}\,\dfrac{\tilde{v}_{j+1} - \tilde{v}_{j-1}}{2h}\,h \\ \qquad\qquad + 2\,\dfrac{d}{dt}\,(\tilde{u}_0\,\tilde{v}_0) = 0 \\ \forall\ \tilde{v}_h = (\tilde{v}_j) \,/\, \displaystyle\sum_j \tilde{v}_j^2\,h < +\infty \end{array}\right.$$

Comparing (5.6) with the continuous variational formulation (1.10) we gave in section 1, we see that:

- the bilinear form $a(u,v) = \displaystyle\int_{\mathbb{R}} \frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\,dx$ has been approximated by :

$$(5.7)\qquad a_h(u_h,v_h) = \frac{4}{3}\sum_j \frac{u_{j+1} - u_j}{h}\,\frac{v_{j+1} - v_j}{h}\,h - \frac{1}{3}\sum_j \frac{u_{j+1} - u_{j-1}}{2h}\,\frac{v_{j+1} - v_{j-1}}{2h}\,h$$

while te bilinear form $b(u,v) = 2\,u(0)\,v(0)$ has been replaced by :

$$(5.8)\qquad b_h(u_h,v_h) = 2\,u_0\,v_0$$

It is easy to derive the discrete energy identity :

(5.9) $$\frac{d}{dt}\left\{\frac{1}{2}\left\|\frac{du_h}{dt}\right\|^2+\frac{1}{2}\, a_h(u_h,u_h)\right\}+b_h\left(\frac{du_h}{dt},\frac{du_h}{dt}\right)=0$$

which is the equivalent of (1.12) (we have set $\|u_h\|^2=\sum_j |u_j|^2 h$). As $b_h(.,.)$ is positive, this means that the discrete energy :

(5.10) $$E_h(t)=\frac{1}{2}\left\|\frac{du_h}{dt}\right\|^2+\frac{1}{2}\, a_h(u_h,u_h)$$

is a decreasing function of time. This proves the L^2-stability of the semi discrete problem (5;5), since one has the discrete coerciveness inequality (see [2] for instance) :

(5.11) $$a_h(u_h,u_h)\geq\sum_j\left|\frac{u_{j+1}-u_h}{h}\right|^2 h$$

Now to get our discrete problem in the half line $x<0$ we use that $\tilde{u}_h$ is even with respect to the index j and that the approximate solution u_h we are looking for is the restriction of $\tilde{u}_h$ to the negative indices, according to the "algorithm" defined in section 1. This means that we take :

(5.12) $$\begin{cases} u_j=\tilde{u}_j & j\leq 0\\ u_{-1}=\tilde{u}_1 & \\ u_{-2}=\tilde{u}_2 & \end{cases}$$

The resulting set of equations for the functions $\{u_j, j\leq 0\}$ is the following one :

(5.13) $$\begin{cases} \dfrac{d^2u_j}{dt^2}-\dfrac{4}{3}\,\dfrac{u_{j+1}-2u_j+u_{j-1}}{h^2}+\dfrac{1}{3}\,\dfrac{u_{j+2}-2u_j+u_{j-2}}{4h^2}=0 & j\leq -2\\ \dfrac{d^2u_{-1}}{dt^2}-\dfrac{4}{3}\,\dfrac{u_0-2u_{-1}+u_{-2}}{h^2}+\dfrac{1}{3}\,\dfrac{u_{-3}-u_{-1}}{2h^2}=0 & j=-1\\ \dfrac{d^2u_0}{dt^2}-\dfrac{8}{3}\,\dfrac{u_{-1}-u_0}{h^2}+\dfrac{2}{3}\,\dfrac{u_{-2}-u_0}{4h^2}+\dfrac{2}{h}\,\dfrac{du_0}{dt}=0 & j=0 \end{cases}$$

Of course, the L^2-stability of problem (5.13) stems up from the one of problem (5.5).

Let us go now to the time discretization. When the wave equation is posed in the whole space, a fourth order scheme in space and time is given by ([2],[12]) :

(5.14) $$\begin{aligned} &\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2}-\frac{4}{3}\,\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}+\frac{1}{3}\,\frac{u_{j+2}^n-2u_j^n+u_{j-2}^n}{4h^2}\\ &\qquad -\frac{\Delta t^2}{12}\,\frac{u_{j+2}^n-4u_{j+1}^n+6u_j^n-4u_{j-1}^n+u_{j-2}^n}{h^4}=0 \end{aligned}$$

where u_j^n is the approximation of $u(x_j,t^n)$, $x_j = jh$, $t^n = n\Delta t$. In (5.14), the last term is added to achieve the fourth order accuracy with respect to time. The L^2-stability of (5.14) derives from the conservation of the discrete energy :

$$
(5.15)\quad \left|\begin{aligned} E_h^{n+1/2} &= \frac{1}{2}\left\|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right\|^2 + \frac{1}{2}\, a_h\left(u_h^{n+1},u_h^n\right) \\ &- \frac{\Delta t^2}{12}\sum_j \frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2}\,\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}\,h \end{aligned}\right.
$$

which is proved to be a positive quadratic form (see [2]) under the stability condition $\frac{\Delta t}{h} \le 1$.
The simplest way to construct a stable approximation for the modified problem (5.3) in $\tilde{u}$ which coincides with the fourth order scheme (5.14) when $j \neq 0$, consists in considering the following equations :

$$
(5.16)\quad \left|\begin{aligned} &\frac{\tilde{u}_j^{n+1}-2\tilde{u}_j^n+\tilde{u}_j^{n-1}}{\Delta t^2} - \frac{4}{3}\,\frac{\tilde{u}_{j+1}^{n+1}-2\tilde{u}_j^n+\tilde{u}_{j-1}^n}{h^2} + \frac{1}{3}\,\frac{\tilde{u}_{j+2}^n-2\tilde{u}_j^n+\tilde{u}_{j-2}^n}{4h^2} \\ &\qquad - \frac{\Delta t^2}{12}\,\frac{\tilde{u}_{j+2}^n-4\tilde{u}_{j+1}^n+6\tilde{u}_j^n-4\tilde{u}_{j-1}^n+\tilde{u}_{j-2}^n}{h^4} = 0 \qquad\qquad j \neq 0 \\ &\frac{\tilde{u}_0^{n+1}-2\tilde{u}_0^n+\tilde{u}_0^{n-1}}{\Delta t^2} - \frac{4}{3}\,\frac{\tilde{u}_1^n-2\tilde{u}_0^n+\tilde{u}_{-1}^n}{h^2} + \frac{1}{3}\,\frac{\tilde{u}_2^n-2\tilde{u}_0^n+\tilde{u}_{-2}^n}{4h^2} \\ &\qquad + \frac{2}{h}\,\frac{\tilde{u}_0^{n+1}-\tilde{u}_0^{n-1}}{2\Delta t} - \frac{\Delta t^2}{12}\,\frac{u_2^n-4u_1^n+6u_0^n-4u_{-1}^n+u_{-2}^n}{h^4} = 0 \qquad j = 0 \end{aligned}\right.
$$

It is easy to see that the solution $\tilde{u}_h^n$ satisfies the following identity :

$$
(5.17)\quad \frac{1}{\Delta t}\left\{\tilde{E}_h^{n+1}-\tilde{E}_h^n\right\} + 2\left|\frac{\tilde{u}_0^{n+1}-\tilde{u}_0^{n-1}}{\Delta t}\right|^2 = 0
$$

This proves the decay of the discrete energy $\tilde{E}_h^{n+1/2}$ with respect to n, $\tilde{E}_h^{n+1/2}$ being defined as $E_h^{n+1/2}$ by replacing u_h^n by $\tilde{u}_h^n$, and therefore the L^2-stability of problem (5.16) is obtained as soon as one satisfies the stability condition :

$$
(5.18)\quad \frac{\Delta t}{h} \le 1
$$

To write now the scheme for the approximate solution u_h^n in the half-line $x < 0$, we adopt the same approach as in the semi-discrete case. We obtain the following equations :

$$
(5.19)\qquad
\left|\begin{array}{ll}
\dfrac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - \dfrac{4}{3}\,\dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + \dfrac{1}{3}\,\dfrac{u_{j+2}^n - 2u_j^n + u_{j-2}^n}{4h^2} & \\
\qquad - \dfrac{\Delta t^2}{12}\,\dfrac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n}{h^4} = 0 & j \leq -2 \\
\dfrac{u_{-1}^{n+1} - 2u_{-1}^n + u_{-1}^{n-1}}{\Delta t^2} - \dfrac{4}{3}\,\dfrac{u_0^n - 2u_{-1}^n + u_{-2}^n}{h^2} + \dfrac{1}{3}\,\dfrac{u_{-3}^n - u_{-1}^n}{2h^2} & \\
\qquad - \dfrac{\Delta t^2}{12}\,\dfrac{7u_{-1}^n - 4u_0^n - 4u_{-2}^n + u_{-3}^n}{h^4} = 0 & j = -1 \\
\dfrac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{\Delta t^2} - \dfrac{8}{3}\,\dfrac{u_{-1}^n - u_0^n}{h^2} + \dfrac{2}{3}\,\dfrac{u_{-2}^n - u_0^n}{4h^2} & \\
\qquad + \dfrac{2}{h}\,\dfrac{u_0^{n+1} - u_0^{n-1}}{2\Delta t} - \dfrac{\Delta t^2}{12}\,\dfrac{6u_0^n - 8u_{-1}^n + 2u_{-2}^n}{h^4} = 0 & j = 0
\end{array}\right.
$$

We obtain a numerical scheme which has the following properties :

- it is completely explicit (and thus very easy to implement)
- it is L^2-stable under the stability condition $\frac{\Delta t}{h} \leq 1$
- it gives a fourth order approximation of the wave equation in the interior domain $x < 0$
- it is consistent, via the image principle, with the transparent boundary condition at $x = 0$.

It is a fact that using this scheme, we loose the fourth order accuracy at the boundary. Nevertheless, such a drawback is not really troublesome since the higher order error term will affect a part of the solution, namely the reflected wave, which is supposed to be itself very small. This prevision is, anyway, confirmed by the numerical results we are going to present. Of course, it should be possible to get an higher accuracy first by taking a best space approximation of the δ-function than the one we get with the function $\delta_h(x)$ we have chosen here. Unfortunately, such an approximation would require an approximate function of $\delta(x)$ with a non constant sign (in order that the different moments of this approximate function vanish up to a certain order) so that the similar energy arguments would be more involved. We prefer to delay this question later and to conclude this section by showing some numerical results made with the scheme (5.19). Let us consider the following 1-d Cauchy problem:

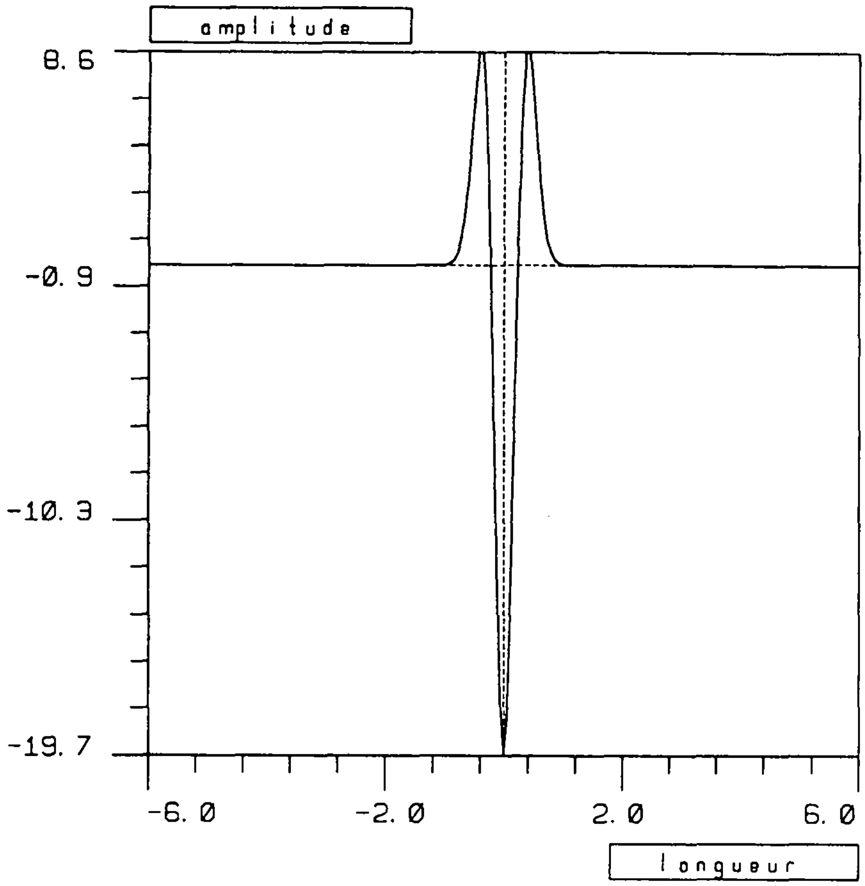
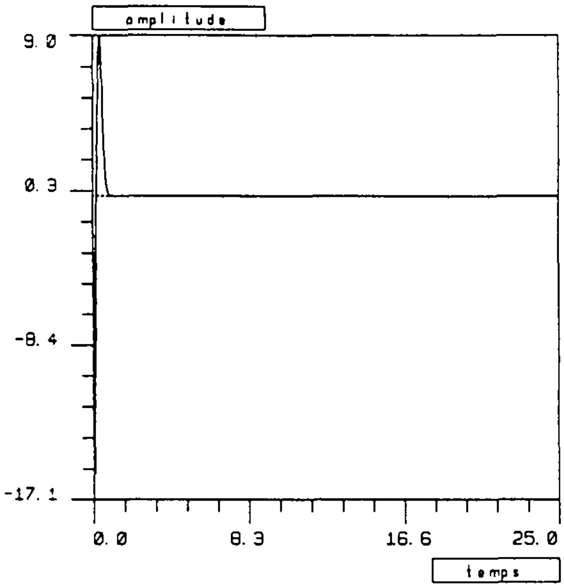
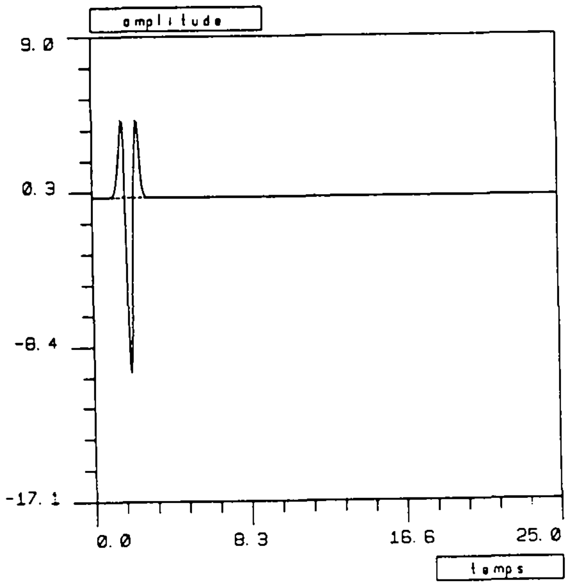


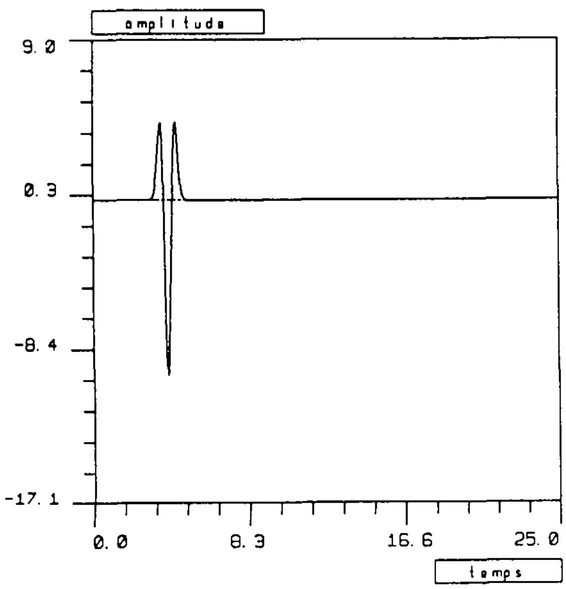
Fig. 3-1. : The initial data u_0.



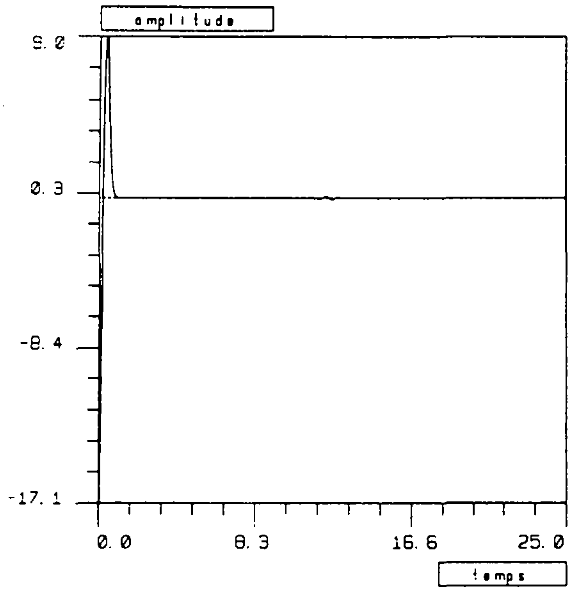
Solution at x=0
Exact solution
Fig. 3-2.



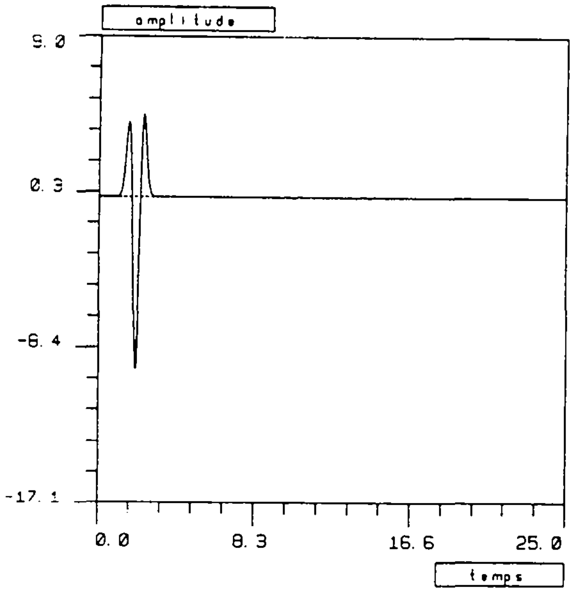
Solution at x=2
Exact solution
Fig. 3-3.



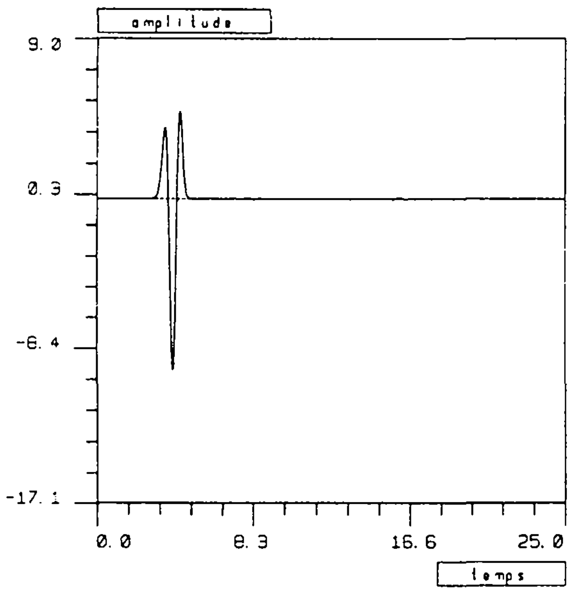
Solution at x=4
Exact solution
Fig. 3-4.



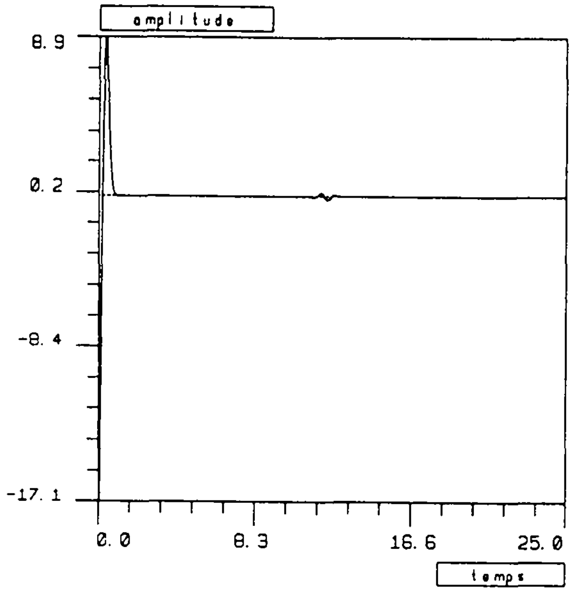
Solution at x=0
Second order scheme
Fig. 3-5.



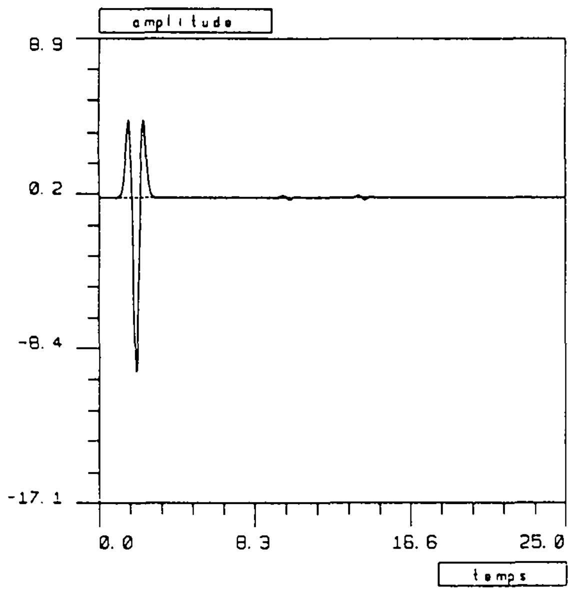
Solution at x=2
Second order scheme
Fig. 3-6.



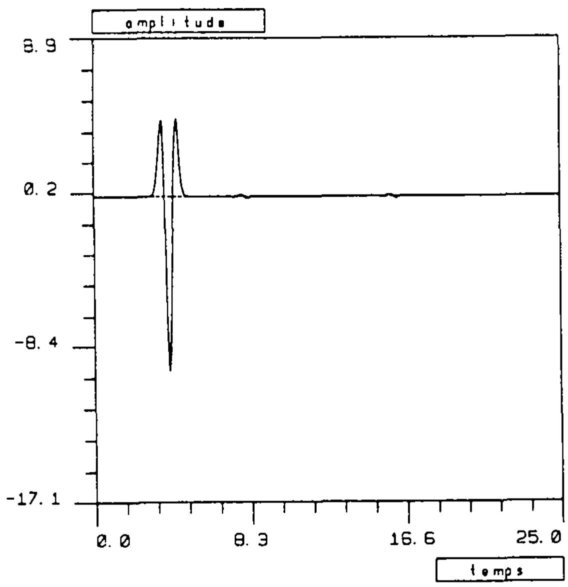
Solution at x=4
Second order scheme
Fig. 3-7.



Solution at x=0
Fourth order scheme
Fig. 3-8.



Solution at x=2
Fourth order scheme
Fig. 3-9.



Solution at x=4
Fourth order scheme
Fig. 3-10.

5.2 The second order condition in the higher dimensional case.

There is no difference in the discretization problem due to the dimension as far as the first order absorbing boundary condition is concerned. So, in the multidimensional case, we deal directly with the second order condition. Moreover, we shall restrict ourselves to the case of dimension d=2, the differences with the higher dimension case are only minor .

As in the 1-d case, let us consider first the spatial discretization. With a one-point discrete Dirac, the second order equation of images (4.1) is discretized as:

(5.20) $$\frac{d^2u_{i,j}}{dt^2} - (\Delta_h u_h)_{i,j} + \frac{2}{h}\delta_o^i R_j = 0$$

where

(5.21) $$u_h = (u_{i,j})_{(i,j)\in Z^2}$$

$$\Delta_h u_h = \Delta_{x,h}u_h + \Delta_{y,h}u_h$$

$$(\Delta_{x,h}u_h)_{i,j} = \frac{4}{3}\,\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} - \frac{1}{3}\,\frac{u_{i+2,j}-2u_{i,j}+u_{i-2,j}}{4h^2}$$

$$(\Delta_{y,h}u_h)_{i,j} = \frac{4}{3}\,\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2} - \frac{1}{3}\,\frac{u_{i,j+2}-2u_{i,j}+u_{i,j-2}}{4h^2}$$

and δ_k^i the Kronecker symbol.

The term R_j should be an approximation of Bu(o,jh) where B is given by (4.2). Thus, we should look for an approximation of the form:

((5.22) $$\left\{\begin{array}{l} R_j = \dfrac{du_{0,j}}{dt} - \beta\varphi_j \\ \dot{\varphi}_j = (\tilde{\Delta}_{y,h}u_h)_{0,j} \end{array}\right.$$

where $\tilde{\Delta}_{y,h}$ is an approximation of Δ_Γ , which can naturally be equal to $\Delta_{y,h}$ as defined by (5.21). In this case, eliminating φ_j in (5.22) by using (5.20), one gets:

(5.23) $$\dot{R}_j = (\Delta_{x,h}u_h)_{0,j} + (1-\beta)(\Delta_{y,h}u_h)_{0,j} - \frac{2}{h}R_j$$

This corresponds to the following discretization of the operator-valued matrix A of (4.3):

$$A \cong A_h = \begin{pmatrix} 0 & -I & 0 \\ -\Delta_h & \frac{2}{h}\delta_0^i & -\frac{2\beta}{h}\delta_0^i \\ \Delta_{y,h}\gamma & 0 & 0 \end{pmatrix}$$

where $(\gamma u_h)_j = u_{0,j}$.

However, we are not able to obtain, from this discretization of the boundary term Bu, a discrete equivalent of the energy formula (4.16). The reason of this defect lies in an extra term that can not be avoided when one tries to establish a discrete Green formula for the fourth order discrete laplacian Δ_h. An alternate strategy, which arises from the above analysis, is to deal with R_j directly, instead of going through φ_j We are then not restricted to use (5.20) to calculate $\frac{d^2u_{0,j}}{dt^2}$.
Actually, we shall substitute (5.23) by

$$\dot{R}_j = (1-\beta)(\Delta_{x,h}u_h)_{0,j}+\beta\frac{u_{-2,j}-2u_{0,j}+u_{2,j}}{4h^2} + (1-\beta)(\Delta_{y,h}u_h)_{0,j} - \frac{2}{h}R_j \tag{5.24}$$

Note that, if the variable $\widetilde{U} = (u,v,R)$ is used in the place of $U = (u,v,\varphi)$, than we can write (4.1) as an evolution equation, with the following operator valued matrix $\widetilde{A}$ instead of A in (4.3):

$$\widetilde{A} = \begin{pmatrix} 0 & -I & 0 \\ -\Delta & 0 & 2\delta_\Gamma \\ -\gamma((1-\beta)\Delta+\beta\frac{\partial^2}{\partial x^2}) & 0 & \gamma(2\delta_\Gamma) \end{pmatrix} \tag{5.25}$$

Then (5.20 and 5.24) correspond to the following, clearly consistent discretization of $\widetilde{A}$:

(5.26) $$\widetilde{A_h} = \begin{pmatrix} 0 & -I & 0 \\ -\Delta_h & 0 & \frac{2}{h}\delta_0^i \\ -\gamma((1-\beta)\Delta_h+\beta D_{x,2h}) & 0 & \frac{2}{h} \end{pmatrix}$$

where

$$(D_{x,h}u_h)_{i,j}= \frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}$$

With this discretization, we get the following energy formula:

(5.27) $$\frac{d}{dt}\left\{\beta\left[E_h(w_h) - \frac{1}{4}\Sigma(w_{-1,j})^2 + \Sigma(R_j+\frac{1}{2}w_{-1,j})^2\right] +(1-\beta)E_h(v_h)\right\} = -2\Sigma h\, \dot{R}_j^2$$

where (w_h) is the central finite differencing of $\frac{\partial u}{\partial x}$:

$$w_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

It is easy to show that the difference $E_h(w_h) - \frac{1}{2}\Sigma w^2_{-1,j}$ remains a sum of positive terms, representing an energy of w_h. On the other hand, since R_j represente in (5.20) an approximation of Bu(0,jh), the formula (5.27) is what we are looking for : a discrete equivalent of (4.16).
Finally, the time discretization of (5.20) and (5.24) follows the same ideas as in the 1-d case, and we get a stable scheme under the CFL condition $(\frac{\Delta t}{h})^2 < \frac{1}{2}$. Details of these calculations, as well as numerical experiments with the second order absorbing boundary condition will be reported later.

6. CONCLUSIONS AND PERSPECTIVES

We have developed in this paper a very simple and rather general theory which allows one to treat absorbing boundary conditions for the acoustic wave equation by means of a reflection principle analogous to the classical principle of images for Neumann or Dirichlet boundary conditions.This technique permits us to treat the discretization of the problem in a very natural way which does not depend on the order of the scheme one considers for the interior equation. Moreover, it guaratees the stability of the resulting numerical method The weakness of our approach is that it seems difficult to keep the same accuracy for the approximation of the boundary conditions as for the interior equation. It would be interesting to investigate this open question. Let us mention that our approach establishes a quantitative link between the theory of absorbing boundary conditions and the method of absorbing layers, which gives a possible direction for further improvements of the present study.

The main property of the wave equation used for establishing our generalized principle of images is its invariance under the change of variables x --> -x. Therfore, our approach is generalizable to hyperbolic equations or systems possesing this property, as do Maxwell's equations for instance. For the elastodynamic equations, the situation is not so clear since one knows that even with the classical Dirichlet or free boundary conditions, no simple theory of images is available. Nevertheless, we think that it would be possibe to generalize the ideas developed in this paper to elastic waves.

REFERENCES :

[1]. BAYLISS A., JORDAN K. E., LEMESURIER B.J., TURKEL E., *A fourth order accurate finite difference scheme for the computation of elastic waves.* Bull. Seism. Soc. Am. n°4, 1986, pp.1115-1132.

[2] COHEN G., JOLY P., *Fourth order schemes in time and space for heterogeneous wave equations.* (to appear in Comput. Meth. in Appl. Mech. and Eng.)

[3] COURANT R., HILBERT D., Methods of Mathematical Physics, vol.2. Interscience, New York 1962.

[4] DABLAIN M., *The application of high order differencing for the scalar wave equation.* Geophysics, vol 51, n°1, 1986, pp.54-66.

[5] ENGQUIST B., MAJDA A., *Radiation boundary conditions for acoutic and elastic wave calculation.* Comm. Pure Appl. Math., v.32, 1979, pp. 313-357.

[6] HADUONG T., JOLY P., *On the stability analysis of boundary conditions of the wave equation by energy methods.* (in preparation).

[7] HIGDON R.L., *Initial boudary value problems for linear hyperbolic systems.* SIAM Rev., v.28, 1986, pp. 177-217.

[8] ISRAELI M., ORSZAG S., *Approximations of radiation boundary conditions.* Journ. of Comp. Phys., v.41, n°1, 1981, pp. 115-135.

[9] LIONS J.L., MAGENES E., Problèmes aux limites non homogènes et applications. Dunod, Paris 1968.

[10] OLIGER J., *Fourth order difference methods for the initial boundary value problems for hyperbolic equations.* Math. of Comp., v28, 1974, pp. 15-25.

[11] PAZY. A., Semigroups of linear operators and applications to partial differential equations. Springer-verlag, Berlin 1983.

[12] SHUBIN G.R., BELL J.B., *A modified equation approach to constructing fourth order methods for acoustic wave propagation*. SIAM J. Sci. Sta. Comp., v.8, n°2, 1987, pp. 135-151.

[13] TREFETHEN L.N., HALPERN L., *Well-posedness of one way wave equations and absorbing boundary conditions*. Math. of Comp., v.47, 1986, pp. 421-435.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

ISSN 0249-6399